

AB FRQ #1

Thursday, May 06, 2021 11:59 AM

$r f(r)$	0	2	12	25	72
r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

1. The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f , where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

- (a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

$$f'(2.25) \approx \frac{f(2.5) - f(2)}{2.5 - 2} = \frac{10 - 6}{2.5 - 2} = \frac{4}{.5} = 8 \frac{\text{mg}}{\text{cm}^2}$$

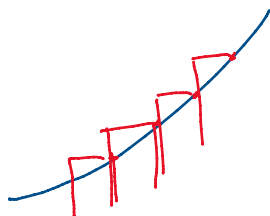
the rate ^{per radius} at which the density of bacteria is increasing at $8 \frac{\text{mg}}{\text{cm}^2} / \text{cm}$

- (b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression

$2\pi \int_0^4 r f(r) dr$. Approximate the value of $2\pi \int_0^4 r f(r) dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

$$2\pi [1 \times 2 + 1 \times 12 + 0.5 \times 25 + 1.5 \times 72] = 845.088 \text{ mg}$$

- (c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.



overestimate because $r f(r)$ is increasing.

- (d) The density of bacteria in the petri dish, for $1 \leq r \leq 4$, is modeled by the function g defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of k , $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \leq r \leq 4$?

$$\text{Average value} = \frac{\int_1^4 g(r) dr}{4 - 1} = g(k)$$

$$= 9.876 = g(k)$$

$$\boxed{k = 2.497 \text{ cm}}$$

AB FRQ #2

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2. A particle, P , is moving along the x -axis. The velocity of particle P at time t is given by $v_P(t) = \sin(t^{1.5})$ for $0 \leq t \leq \pi$. At time $t = 0$, particle P is at position $x = 5$. $x_P(0) = 5$

A second particle, Q , also moves along the x -axis. The velocity of particle Q at time t is given by $v_Q(t) = (t - 1.8) \cdot 1.25^t$ for $0 \leq t \leq \pi$. At time $t = 0$, particle Q is at position $x = 10$. $x_Q(0) = 10$

- (a) Find the positions of particles P and Q at time $t = 1$.

$$x_P(1) = x_P(0) + \int_0^1 v_P(t) dt = 5.371$$

$$x_Q(1) = x_Q(0) + \int_0^1 v_Q(t) dt = 8.564$$

- (b) Are particles P and Q moving toward each other or away from each other at time $t = 1$? Explain your reasoning.

$$v_P(1) = 0.841 \Rightarrow \text{moving right}$$

$$v_Q(1) = -1 \Rightarrow \text{moving left}$$

$$x_Q(1) > x_P(1) \quad Q \text{ is on the right}$$

moving towards each other

- (c) Find the acceleration of particle Q at time $t = 1$. Is the speed of particle Q increasing or decreasing at time $t = 1$? Explain your reasoning.

$$\left. \frac{d(v_Q)}{dt} \right|_{t=1} = 1.027$$

$$a_Q(1) > 0$$

$$v_Q(1) < 0$$

different signs \Rightarrow slowing down

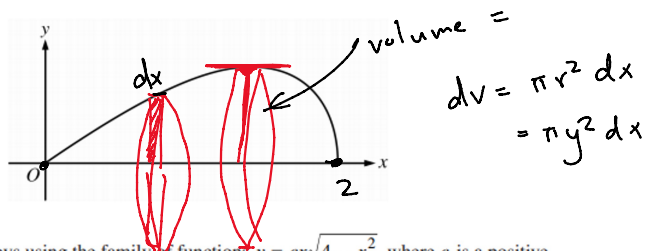
- (d) Find the total distance traveled by particle P over the time interval $0 \leq t \leq \pi$.

π

$$\int_0^1 |v_p(t)| dt = 1.931$$

AB FRQ #3

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3. A company designs spinning toys using the family of functions $y = c\sqrt{4-x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x -axis and the graph of $y = c\sqrt{4-x^2}$, for some c . Each spinning toy is in the shape of the solid generated when such a region is revolved about the x -axis. Both x and y are measured in inches.

- (a) Find the area of the region in the first quadrant bounded by the x -axis and the graph of $y = c\sqrt{4-x^2}$ for $c = 6$.

$$y = 6\sqrt{4-x^2} = 0$$

$$x = 0 \quad 4-x^2 = 0$$

$$x = \pm 2$$

$$A = \int_0^2 6\sqrt{4-x^2} dx$$

$$= 6 \int_{-2}^2 \sqrt{u} du$$

$$= -3 \int u^{1/2} du = -3 \frac{u^{3/2}}{3/2} = -3 \left(\frac{2}{3}\right) u^{3/2}$$

$$= -2 u^{3/2} \Big|_0^4$$

$$2 u^{3/2} \Big|_0^4 = 2(4)^{3/2} = 16$$

- (b) It is known that, for $y = c\sqrt{4-x^2}$, $\frac{dy}{dx} = \frac{c(4-2x^2)}{\sqrt{4-x^2}}$. For a particular spinning toy, the radius of the

largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

$$\frac{dy}{dx} = 0 = \frac{c(4-2x^2)}{\sqrt{4-x^2}} \Rightarrow 4-2x^2 = 0$$

$$4 = 2x^2$$

$$2 = x^2$$

$$x = \pm \sqrt{2}$$

$$\sqrt{2}$$

$$y = \text{radius}$$

$$= c\sqrt{2}\sqrt{4-\sqrt{2}^2} = 1.2$$

$$= c\sqrt{2}\sqrt{2} = 1.2$$

$$2c = 1.2$$

$$c = 0.6$$

- (c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

$$dV = \pi y^2 dx = \pi (c^2 x^2 (4-x^2)) dx$$

$$2$$

$$dV = \pi y^2 dx = \pi (c^2 x^2 (4 - x^2)) dx$$

$$c^2 \pi \int_0^2 x^2 (4 - x^2) dx = c^2 \pi \int_0^2 (4x^2 - x^4) dx$$

$$= c^2 \pi \left(\frac{4}{3} x^3 - \frac{1}{5} x^5 \right) \Big|_{x=0}^2$$

$$= c^2 \pi \left(\frac{32}{3} - \frac{32}{5} \right) \quad \frac{5}{15} - \frac{3}{15} = \frac{2}{15}$$

$$= 32c^2 \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

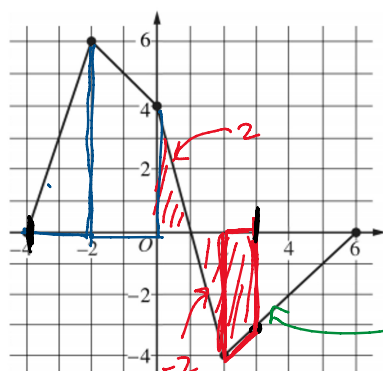
$$= 32c^2 \pi \left(\frac{2}{15} \right) = \frac{64c^2 \pi}{15} = 1$$

$$c^2 = \frac{15}{32}$$

$$c = \sqrt{\frac{15}{32}} = \frac{\sqrt{15}}{4\sqrt{2}} = \frac{\sqrt{30}}{8}$$

AB FRQ #4

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Graph of f

$$G(-4) = \int_0^{-4} f(t) dt$$

$$= - \int_{-4}^0 f(t) dt = -(6 + 10) = -16$$

$$- \frac{1}{2} (4 + 3) (1) = -\frac{7}{2}$$

4. Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f , consisting of four

line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t) dt$.

$$\Rightarrow G'(x) = f(x)$$

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

$$G''(x) > 0$$

$$G'(x) = f(x)$$

$$G'(x) = f'(x) > 0$$

↑
slopes of $f > 0$

$$(-4, -2), (2, 6)$$

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

$$P'(x) = G'(x) f(x) + G(x) f'(x)$$

$$= [f(x)]^2 + G(x) f'(x)$$

$$f(3) = -3$$

$$G(3) = \int_0^3 f(t) dt = \text{area under } f \text{ from } 0 \text{ to } 3 = -\frac{7}{2}$$

$$f'(3) = \text{slope of } f \text{ at } x=3 \Rightarrow 1$$

$$P'(3) = (-3)^2 + -\frac{7}{2}(1) = 9 - \frac{7}{2} = \frac{11}{2}$$

(c) Find $\lim_{x \rightarrow 2} \frac{G(x)}{x^2 - 2x}$.

$$\frac{G(2)}{2^2 - 2 \cdot 2} = \frac{0}{0} \quad \text{indeterminant form}$$

$$\text{use } L$$

$$= \lim_{x \rightarrow 2} \frac{G'(x)}{2x - 2} = \lim_{x \rightarrow 2} \frac{f(x)}{2x - 2} = \frac{-4}{2} = -2$$

$$\stackrel{\text{use L}}{=} \lim_{x \rightarrow 2} \frac{G'(x)}{2x-2} = \lim_{x \rightarrow 2} \frac{f(x)}{2x-2} = \frac{-4}{2} = \boxed{-2}$$

- (d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c , $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

$$\frac{G(2) - G(-4)}{2 - (-4)} = \frac{0 - 16}{6} = \frac{16}{6} = \boxed{\frac{8}{3}}$$

$G'(x) = f(x) \Rightarrow \text{exists} \Rightarrow G'(x) \text{ is differentiable over } [-4, 2]$

differentiable \Rightarrow continuous $[-4, 2]$

\hookrightarrow yes MVT applies b/c $G(x)$ is both differentiable & continuous

AB FRQ #5

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5. Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$.

$$\frac{dy}{dx} = \frac{y \cos x}{4y - \sin x}$$

$$4y \cdot \frac{dy}{dx} = y \cos x + \frac{dy}{dx} \sin x$$

$$4y \frac{dy}{dx} - \frac{dy}{dx} \sin x = y \cos x$$

$$\frac{dy}{dx} (4y - \sin x) = y \cos x$$

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$. $\leftarrow (x_0, y_0)$

$$y - y_0 = m(x - x_0)$$

$$m = \left. \frac{dy}{dx} \right|_{(0, \sqrt{3})} = \frac{\sqrt{3} \cos 0}{4\sqrt{3} - \sin 0} = \frac{1}{4}$$

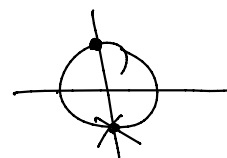
$$y - \sqrt{3} = \frac{1}{4}x$$

(c) For $0 \leq x \leq \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

$$x = \frac{\pi}{2} \rightarrow \frac{dy}{dx} = 0 = \frac{y \cos x}{4y - \sin x} \Rightarrow y \cos x = 0$$

$$y \neq 0 \text{ or } \cos x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$



$$2y^2 - 6 = y \sin x$$

$$2y^2 - y - 6 = 0$$

$$(2y+3)(y-2) = 0$$

$$y = -\frac{3}{2}, y = 2$$

$$\left(\frac{\pi}{2}, 2 \right)$$

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

$$\frac{d^2y}{dx^2} = \frac{(4y - \sin x) \left(\frac{dy}{dx} \cos x - y \sin x \right) - y \cos x \left(4 \frac{dy}{dx} - \cos x \right)}{(4y - \sin x)^2}$$

$$= \frac{(4y - 1)(-1)}{(4y - 1)^2}$$

$$y = 2$$

$$\frac{d^2y}{dx^2} < 0$$

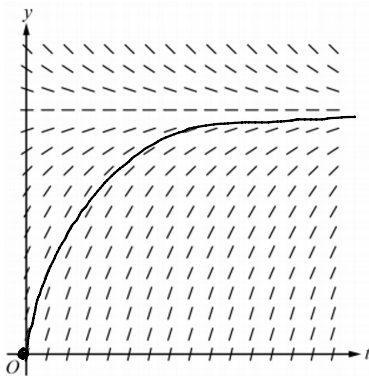
$$\left(\frac{\pi}{2}, 2\right) = \text{relative max}$$

AB FRQ #6

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6. A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

- (a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.



- (b) Using correct units, interpret the statement $\lim_{t \rightarrow \infty} A(t) = 12$ in the context of this problem.

a total of 12mg of medicine is in the patient's body after enough time

- (c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation

$$\frac{dy}{dt} = \frac{12 - y}{3} \text{ with initial condition } A(0) = 0.$$

$$dy = \frac{12 - y}{3} dt$$

$$t = 0 \quad y = 0$$

$$\int \frac{dy}{12 - y} = \int \frac{1}{3} dt$$

$$-\ln(12 - y) = \frac{1}{3}t + C$$

$$\ln(12 - y) = -\frac{1}{3}t + C$$

$$12 - y = e^{-\frac{1}{3}t + C}$$

$$= Ce^{-\frac{1}{3}t}$$

$$y = 12 - Ce^{-\frac{1}{3}t}$$

$$0 = 12 - Ce^0 = 12 - C$$

$$C = 12$$

$$\boxed{y = 12 - 12e^{-\frac{1}{3}t}}$$

$$\ln(12-y) = 3 - \dots = \dots$$

$$12-y = e^{-\frac{1}{3}t+C}$$

$$\boxed{y = 12 - 12e^{-\frac{1}{3}t}}$$

(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams,

of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the

differential equation $\frac{dy}{dt} = 3 - \frac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in

the second patient. Is the rate of change of the amount of medication in the second patient increasing or

decreasing at time $t = 1$? Give a reason for your answer.

$$\text{rate of change} = \frac{dy}{dt} = 3 - \frac{y}{t+2}$$

der

$$\frac{d^2y}{dt^2} = - \left(\frac{(t+2)\frac{dy}{dt} - y \cdot 1}{(t+2)^2} \right)$$

$$= \frac{-(t+2)\frac{dy}{dt} + y}{(t+2)^2} \bigg|_{t=1} = \frac{-(3)\left(\frac{13}{6}\right) + 2.5}{(3)^2} < 0$$

$$t=1$$

$$y=2.5$$

$$\frac{dy}{dt} = 3 - \frac{2.5}{3} = 3 - \frac{5}{6} = \frac{13}{6}$$

$$\boxed{\frac{dy}{dt} \text{ is decreasing}}$$

BC FRQ #2

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2. For time $t \geq 0$, a particle moves in the xy -plane with position $(x(t), y(t))$ and velocity vector

$\langle (t-1)e^{t^2}, \sin(t^{1.25}) \rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

- (a) Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.

$$|v(t)| = \sqrt{[(t-1)e^{t^2}]^2 + \sin^2(t^{1.25})} = 1.271$$

$$a(t) = v'(t)$$

$$a(1.2) = v'(1.2) = \langle 4.247, .405 \rangle$$

- (b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.

$$\int_0^{1.2} |v(t)| dt = 1.610$$

- (c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.


$$x(t) = \min$$

$$\frac{dx(t)}{dt} = v_x(t) = 0$$

$$v_x(t) = (t-1)e^{t^2} = 0$$

$$t = 1$$

t	$0 < t < 1$	$t = 1$	$t > 1$
$v_x(t)$	-	0	+

1 
 $t=1$ is a relative min

$$x(t) - x(0) = \int_0^1 v(t) dt =$$

$$\langle -2, 5 \rangle =$$

$$x(1) = \langle -2, 5 \rangle + \int_0^1 v(t) dt = \boxed{\langle -2.604, 5.410 \rangle}$$

~~no~~ furthest left

$v(t) > 0$ for $t > 1$
 particle is always moving right
 as $t \rightarrow \infty$

BC FRQ #5

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5. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition

$$4 \cdot 1 \ln(1) = 0$$

$f(1) = 4$. It can be shown that $f''(1) = 4$.

$$x=1 \quad y=4$$

- (a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

$$\begin{aligned} T_2(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 4 + \frac{4}{2}(x-1)^2 = \boxed{4 + 2(x-1)^2} \end{aligned}$$

$$f(2) \approx T_2(2) = 4 + 2(2-1)^2 = \boxed{6}$$

- (b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

$$x=2$$

$$dy = y(x \ln x) dx$$

x	y	dx	dy
1	4	0.5	$4(1 \ln 1)(0.5) = 0$
1.5	4	0.5	$4(1.5 \ln 1.5)(0.5) = 3 \ln(1.5)$
2	$4 + 3 \ln 1.5$		

- (c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

condition $f(1) = 4$.

y.

$$dy = y x \ln x dx$$

$$\int \frac{dy}{y} = \int \frac{x \ln x}{x} \frac{dx}{dx}$$

$$u = \ln x \quad v = \frac{1}{2} x^2$$

$$du = \frac{1}{x} dx \quad dv = x dx$$

$$= \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \cdot \frac{1}{x} dx$$

$$e^{\ln|y|} = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C$$

$$|y| = e^{\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C} = C e^{\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2}$$

$$y = C e^{\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2}$$

$$-\frac{1}{4} \Rightarrow C = 4e^{\frac{1}{4}}$$

$$x=1 \quad y=4$$

$$x=1 \quad y=.$$

$$y = Ce^{-}$$

$$4 = Ce^{-\frac{1}{4}} \Rightarrow C = 4e^{\frac{1}{4}}$$

$$y = 4e^{\frac{1}{4}} e^{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2}$$

$$y = 4e^{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \frac{1}{4}}$$

6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3} \text{ on its interval of convergence.}$$

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

$$- \frac{1}{e^n} > 0 \quad \checkmark$$

$$- \frac{1}{e^n} \text{ decreasing } \checkmark$$

$$\int_0^{\infty} \frac{1}{e^x} dx = \int_0^{\infty} e^{-x} dx = \lim_{c \rightarrow \infty} \int_0^c e^{-x} dx = \lim_{c \rightarrow \infty} 1 - e^{-c} = 1$$

converges

$$\int_0^c e^{-x} dx = -e^{-x} \Big|_0^c = -e^{-c} - (-e^0) = 1 - e^{-c}$$

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ \lim

converges absolutely. $\lim_{n \rightarrow \infty} \left| \frac{\frac{1}{2e^n + 3}}{\frac{1}{e^n}} \right| = \lim_{n \rightarrow \infty} \frac{e^n}{2e^n + 3} = \frac{\infty}{\infty}$

use LH

$$\lim_{n \rightarrow \infty} \frac{1e^n}{2e^n} = \frac{1}{2} \quad 0 < \frac{1}{2} < \infty$$

both series converge or diverge
but $\frac{1}{e^n}$ converges

(c) Determine the radius of convergence of the Maclaurin series for g .

$$\sum \frac{(-1)^n x^n}{2e^n + 3} \quad \text{ratio test}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{2e^{n+1} + 3} \cdot \frac{2e^n + 3}{x^n} \right|$$

$$= \lim_{n \rightarrow \infty} |x| \left| \frac{2e^n + 3}{2e^{n+1} + 3} \right|$$

$$= |x| \lim_{n \rightarrow \infty} \frac{2e^n + 3}{2e^{n+1} + 3} = |x| \frac{1}{e} < 1$$

$$|x| < \frac{1}{e}$$

$$-\frac{1}{e} < x < \frac{1}{e}$$

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating

series error bound to determine an upper bound on the error of the approximation.

$$g(1) = \frac{(-1)^0}{2e^0 + 3} + \frac{(-1)^1}{2e^1 + 3} + \frac{(-1)^2}{2e^2 + 3}$$

error bound

$$\text{error} \leq \frac{1}{2e^2 + 3}$$